ON THE SEPARATION OF MAGNETOHYDRODYNAMIC BOUNDARY LAYERS

(OB OTRYVE MAGNITOGIDRODINAMICHESKOGO POGRANICHNOGO SLOIA)

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Approximate relations will be derived connecting the parameters of the external flow, the boundary layer thickness and the electromagnetic quantities at the section where the laminar magnetohydrodynamic boundary layer detaches.

1. Let us consider a two-dimensional boundary layer on the surface of a body or on the wall of a channel. The x-axis will be directed along the wall and the y-axis perpendicular to it. We assume that the external magnetic field vector lies in the xy-plane and that the characteristic dimension Δ of the variation of its components either exceeds or equals, in order of magnitude, the characteristic body length l. Moreover, we assume that the magnetic Reynolds number, defined by the characteristic velocity, the electrical conductivity and the length l, is equal to or less than unity in order of magnitude. Then the equations of the boundary layer for isotropic transport properties may be written as

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -p' + \frac{\partial}{\partial y} \eta \frac{\partial u}{\partial y} - \frac{\sigma}{c} EB - \frac{u\sigma}{c^2} B^2 \quad (\mathbf{E} = (0, 0, E), \quad E = \mathrm{const}) \tag{1.1}$$

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 \tag{1.2}$$

$$\rho c_p u \frac{\partial T}{\partial x} + \rho c_p v \frac{\partial T}{\partial y} = up' + \eta \left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial}{\partial y} k \frac{\partial T}{\partial y} + \sigma \left(\frac{u^2}{c^2} B^2 + E^2 + \frac{2u}{c} EB\right) \tag{1.3}$$

$$f(p, \rho, T) = 0 \tag{1.4}$$

In the system (1.1) to (1.4), E is the constant z-component of the electric field, $B = B(x)$ is the y-component of the magnetic field vector on the wall, $p' = (dp_0/dx)$ is the pressure gradient in the external flow,

c is the speed of light in vacuum and the rest of the notation has the usual meaning. In the momentum and energy equations, we have omitted terms whose ratios to the remaining terms have the order (δ/l), (δ/Δ) or higher; δ being the boundary layer thickness. Differentiating the momentum and energy equations with respect to y and taking into account the accuracy of the boundary layer theory is equivalent to differentiating equations (1.1) and (1.3) with respect to y.

The instant of breakdown of the ideal flow scheme in the boundary layer (separation of the boundary layer) corresponds to the vanishing at the wall of the quantity $\partial u/\partial y$, calculated from the boundary layer equations. We define the separation point on the wall to be that at which $u = 0$, $v = 0$ and $(\partial u/\partial y) = 0$. To arrive at a relation between the parameters at the separation section (corresponding to the separation point), we shall use a method well-known in hydrodynamics, in which the velocity profile at the separation point is first determined. To this end, we successively differentiate equations (1.1) and (1.3) with respect to y, use the obtained expression for the separation point and compute the derivatives $\partial^2 u/\partial y^2$, $\partial^3 u/\partial y^3$, ..., which permit the velocity profile to be represented in a Taylor series.

2. We assume that the surface of the body is thermally insulated and that the electric field in it is zero. The velocity profile at the point of separation, as described above, can be represented as

$$u(y) = \frac{p' y^2}{2\eta}\left[1 + \frac{\sigma B^2 y^2}{12\eta c^2} + \frac{1}{360}\left(\frac{\sigma B^2 y^2}{\eta c^2}\right)^2 + \frac{A y^4}{360} + \cdots\right] \qquad (2.1)$$

Here the quantities B, σ, A, p' and η, all correspond to the values at the separation point, and the value of A is independent of the magnetic field and the electrical conductivity. Let us estimate the order of magnitude of the term contained in the square brackets. The quantity $(\sigma B^2 \delta^2/\eta c^2) = H^2$, representing the Hartmann number defined by the boundary layer thickness, characterizes the ratio of the magnetic drag to the viscous drag. Since the viscous forces and the inertial forces are of the same order in the boundary layer, the quantity H^2 may also be considered as the ratio of the magnetic forces to the inertial forces, i.e. in order of magnitude it agrees with the magnetic interaction parameter. The latter parameter has order unity in practical applications. Thus, the second term in the square brackets is of the order of 1/12, the third term of the order of 1/360, while the fourth term (for which an order of magnitude estimate had been carried out in studying separation of hydrodynamic boundary layers) is of the order of thousandths. Consequently, (retaining the third term in case the Hartmann number should be significantly greater than one) the velocity profile at the separation point may be approximately expressed as

$$u(y) = \frac{p'y^2}{2\eta}\left[1 + \frac{\sigma B^2 y^2}{12\eta c^2} + \frac{1}{360}\left(\frac{\sigma B^2 y^2}{\eta c^2}\right)^2\right] \tag{2.2}$$

In other words, the separation of the boundary layer occurs at the section where the flow velocity u_0 (at the outer edge), the pressure gradient p', the boundary layer thickness δ and the strength of the magnetic field are connected by the relation

$$\xi^* = 2\left[1 + \frac{\sigma B^2\delta^2}{12\eta c^2} + \frac{1}{360}\left(\frac{\sigma B^2\delta^2}{\eta c^2}\right)^2\right]^{-1} \qquad \left(\xi = \frac{p'\delta^2}{\eta u_0}\right) \tag{2.3}$$

Here ξ^* is the value of ξ at the separation section. For $B = 0$, we obtain from (2.3) the well-known result from hydrodynamics $\xi^* = 2$. In the range $0 \leqslant H \leqslant 4$ of the Hartmann number H, we may write ξ^* as follows with reasonable accuracy (the relative error increases monotonically with H and equals 10% at $H = 4$):

$$\xi^* = \frac{H^2}{\cosh H - 1} = f(H) \qquad \left(H^2 = \frac{\sigma B^2\delta^2}{\eta c^2}\right) \tag{2.4}$$

Formula (2.4) is the desired relation. It has been obtained for arbitrary distribution of the magnetic field on the surface and takes into account the properties of the real gas in thermodynamic equilibrium for a thermally insulated surface. We must remember that the Hartmann number in (2.4) is defined by the quantities B, σ and η at the separation point. For an incompressible fluid with constant σ and η, (2.4) is correct for any thermal range of the surface. As H increases, the function $f(H)$ monotonically decreases. Consequently, for $B \neq 0$, separation occurs at smaller values of ξ^* than for $B = 0$. Compare for example the flow around a body for $B = 0$ and for $B \neq 0$, assuming that the external flow does not depend on the field. Since with the field, $f(H) < f(0) = 2$, and the boundary layer thickness increases faster than for $B = 0$, the separation point moves upstream along the body surface.

If the pressure distribution at the outer edge of the boundary layer is known, then, computing the boundary layer thickness from some empirical formula, we may use (2.4) to determine the separation point of the boundary layer without having to solve equations (1.1) to (1.4).

We note that in (2.1), the second derivative of the pressure first appears in the coefficient for the sixth power of y, while the derivative of the magnetic field dB/dx enters in coefficients of still higher powers of y. This permits us to conclude that the flow at the section where separation occurs depends only weakly on the nature of the external flow and of the magnetic field far away from this section. In the absence of the magnetic field, such a conclusion is usually extended to any arbitrary section [1]. Whether this assertion can be extended to the entire magnetohydrodynamic boundary layer requires additional justifications.

In concluding this section, let us give a confirmation of the approximate formula (2.2). Let us consider the boundary layer in the divergent flow of a conducting fluid between two plane walls forming an angle (diffuser), the magnetic field being such that it is perpendicular to the plane of the flow at the vertex of the angle. This flow has been studied in, e.g. [2]. In particular, an exact solution was obtained for the case where the wall drag vanishes. We may verify that the velocity distribution corresponding to this exact solution agrees reasonably well with that given approximately by (2.2).

3. In order to calculate the magnetohydrodynamic boundary layer and its separation, we may use the Karman integral relations, as in hydrodynamic boundary layers. We shall derive the equation for the thickness of the boundary layer for the case of Prandtl number unity, $E = 0$, and insulated wall. From the energy equation (1.3), we find that the stagnation enthalpy i_* in the boundary layer is constant and is equal to the stagnation enthalpy i_{*0} in the external flow. Let $(u/u_0 = \varphi(x, \delta, y/\delta) = \varphi(x, \delta, z)$. Considering the density, electrical conductivity, and the viscosity coefficient η as functions of the pressure and the enthalpy i, we find

$$\frac{\rho}{\rho^\circ} = \frac{\rho(p, i)}{\rho^\circ} = \frac{1}{\rho^\circ}\rho(p, i_{*0} - 0.5u_0^2\varphi^2) = N_1(x, \delta, z), \qquad \frac{\sigma}{\sigma^\circ} = N_2(x, \delta, z)$$

where ρ° and σ° are characteristic constants. Integrating (1.1) from $y = 0$ to $y = \delta$, we get

$$\frac{d}{dx}\frac{\delta^2}{2} = \frac{\delta^2}{2}\left(\frac{2\alpha_1\sigma^\circ B^2}{c^2\alpha_2\rho^\circ u_0} - \frac{2\alpha_3}{\alpha_2 u_0}\frac{du_0}{dx} - 2\frac{d\ln\alpha_2}{dx} + \frac{2p'}{\rho^\circ u_0^2\alpha_2}\right) + \frac{\eta(p, i_{*0})}{\rho^\circ u_0\alpha_2}\left(\frac{\partial\varphi}{\partial z}\right)_{z=0} \quad (3.1)$$

$$\left(\alpha_1 = \int_0^1 \varphi N_2 dz, \quad \alpha_2 = \int_0^1 (\varphi - \varphi^2) N_1 dz, \quad \alpha_3 = \int_0^1 (\varphi - 2\varphi^2) N_1 dz\right)$$

If the boundary layer thickness does not enter the function $\varphi(x, \delta, z)$, then equation (3.1) becomes a linear first order differential equation.

Equation (3.1) assumes a much simpler form for the case of flat walls Using this, the increase of the rate of growth of the boundary layer thickness, the increase of the total drag and the decrease of the friction drag, when a magnetic field is present, can easily be demonstrated.

4. Consider an incompressible medium with constant σ and η, and a non-vanishing electric field. One easily verifies that a relation analogous to (2.4) can be obtained by substituting $p' + (\sigma EB/c)$ for p'

in (2.4). Thus, we find

$$\frac{p' + (\sigma EB / c)}{\eta u_0} = \frac{H^2}{\cosh H - 1} \tag{4.1}$$

or, using equation (1.1) for the external flow

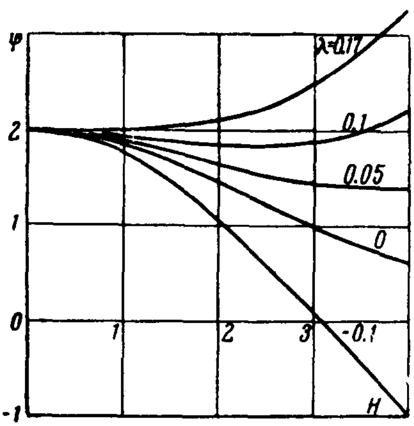
$$-\frac{\delta^2}{\eta u_0}\left(\rho u_0 \frac{du_0}{dx} + \frac{\sigma B^2 u_0}{c^2}\right) = \frac{H^2}{\cosh H - 1} \tag{4.2}$$

Since $f(H) > 0$, in order that separation occurs, the external flow must be a retarding flow, and $|du_0/dx)| > (\sigma B^2/c^2\rho)$.

To determine the value ξ^* at which separation occurs, expression (4.1) may be written as

$$\xi^* = \lambda H^2 + \frac{H^2}{\cosh H - 1} = \psi(\lambda, H) \qquad \left(\lambda = -\frac{c}{u_0 B}\right) \tag{4.3}$$

The term λH^2 reflects the electromagnetic field, which gives a body force $(-\sigma EB/c)$ per unit volume, while the term $H^2(\cosh H - 1)^{-1}$ reflects the magnetohydrodynamic influence, which gives a force $(-\sigma B^2 u/c^2)$ per unit volume. For $\lambda > 0$, the force $(-\sigma EB/c)$ acts in the direction of the flow, while the force $(-\sigma B^2 u/c^2)$ acts against the flow; for $\lambda < 0$, both forces act against the flow. The function $\psi(\lambda, H)$ is given in the figure. The influence of the electromagnetic field on the quantity ξ^* is seen to be greater than the magnetohydrodynamic influence. For small $\lambda > 0$ and sufficiently large H, we have $\xi^* > 2$ (for $\lambda > 1/6$, ξ^* always > 2); while $\xi^* < 2$ for $\lambda = 0$ and any $H \neq 0$. For $\lambda < 0$, we may have $\xi^* < 0$. The explanation of the influence of the electromagnetic field on ξ^* consists of the following. The influence of the force that distorts the profile is first felt at the part of the profile nearest to the wall. The force $(-\sigma B^2 u/c^2)$ for a small velocity (near the wall) is smaller than the force $(-\sigma EB/c)$, which is independent of velocity.



5. Let us consider a simple exact solution, to confirm the conclusion of the previous section. Let an incompressible fluid with constant η move in a plane channel $-\infty < x < +\infty$, $0 < y < \delta$, the upper wall of which has a velocity u_0 = const, and the lower wall is stationary (Couette flow). Let there be a constant electric field along the z-direction, and

a constant magnetic field in the y-direction. For generality, let the electrical conductivity be an arbitrary function of the velocity $\sigma = \sigma^0\chi(u/u_0)$. The flow is described by the equations

$$\varphi'' - H^2\varphi\chi + \lambda H^2\chi - \xi = 0, \qquad \varphi(0) = 0, \qquad \varphi(1) = 1 \tag{5.1}$$

$$\left(\frac{u}{u_0} = \varphi(z),\ z = \frac{y}{\delta},\ \frac{d\varphi}{dz} = \varphi',\ H^2 = \frac{\sigma^0 B^2\delta^2}{c^2\eta},\ \lambda = -\frac{cE}{u_0 B},\ \xi = \frac{p'\delta^2}{u_0\eta}\right)$$

Its solution has the form

$$\frac{y}{\delta} = \int_0^\varphi \frac{d\varphi}{\sqrt{\varphi'(0) + 2q(\varphi)}}, \qquad q(\varphi) = \int_0^\varphi (\xi + H^2\varphi\chi - \lambda H^2\chi)\, d\varphi \tag{5.2}$$

$$1 = \int_0^1 \frac{d\varphi}{\sqrt{\varphi'(0) + 2q(\varphi)}} \tag{5.3}$$

To determine ξ^* from (5.3), it is necessary to set $\varphi'(0) = 0$. If we assume that the electrical conductivity is constant ($\chi = 1$), we find from (5.3) the condition equivalent to (4.3).

6. Let us now consider boundary layers formed on the walls of channel-electrodes. Let the x-axis be in the direction of the electrode and the y-axis perpendicular to it. It is known that the equations of the boundary layer have the form [3]

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -p' + \frac{\partial}{\partial y}\eta\frac{\partial u}{\partial y} - \frac{1}{c} jB \qquad (j_y = j(x),\ B = B(x)) \tag{6.1}$$

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 \tag{6.2}$$

$$\rho c_p u \frac{\partial T}{\partial x} + \rho c_p v \frac{\partial T}{\partial y} = up' + \frac{\partial}{\partial y} k \frac{\partial T}{\partial y} + \eta\left(\frac{\partial u}{\partial y}\right)^2 + \frac{j^2}{\sigma} \tag{6.3}$$

$$f(p, \rho, T) = 0 \tag{6.4}$$

In equations (6.1) and (6.3), j is the component of the current density in the y-direction and B is the component of the magnetic field in the $(-z)$-direction. The magnitude of j is found from the solution of the inviscid flow problem.

For simplicity, we assume that the variation of ρ and η in the boundary layer may be neglected. Then, using the method given in Section 1 and 2, we find the following relations for the separation point:

$$\frac{\delta^2}{\eta u_0}\left(\frac{1}{c} jB + p'\right) = 2, \qquad \xi^* = 2 - \frac{j\delta^2 B}{c\eta u_0}, \qquad -\frac{\rho\delta^2}{\eta}\frac{du_0}{dx} = 2 \tag{6.5}$$

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